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Spring 2021

Stitching Dedekind Cuts to Construct the Real Numbers

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January 25, 2021

If, as the well-known proverb states, "Necessity is the mother of invention," what is born out of *frustration*? For German mathematician, Richard Dedekind,[1] it was apparently a thorough development of the set of real numbers based on the properties of the rational numbers. When Dedekind began teaching in 1858 at the Polytechnic School in Zurich, Switzerland, he was dissatisfied with the way differential calculus was developed and taught in the courses. Although he found the use of geometric intuition useful, even indispensable in introducing the material, he deemed the explanations and proofs presented were not rigorous enough. At the core of his frustration was the lack of a solid mathematical foundation for certain concepts (including the real numbers) that he felt was necessary to prove the most fundamental theorems in calculus. In particular, even though the idea of irrational numbers had been apparent since ancient times, Dedekind felt that their existence and properties had not been rigorously developed purely in arithmetic terms, based only on what is known about the rational numbers. Consequently, Dedekind took it upon himself to write out a rigorous development of the set of real numbers, resulting in a work published in 1872 called *Continuity and Irrational Numbers* [Dedekind, 1872]. An English translation appeared in 1901 as part of a larger compilation of Dedekind's writings called *Essays on the Theory of Numbers* [Dedekind, 1901].

Continuity and Irrational Numbers consists of seven sections. Section I provides a brief recollection of the primary basic arithmetic and order properties of the set of rational numbers. In Section II, Dedekind next described an analogy between the set of rational numbers and the points of a straight line translating the order relationship between two rational numbers to a left-to-right arrangement of the corresponding points. He also described how [Dedekind, 1901, pp. 7–8]

> this analogy between rational numbers and the points of a straight line ... becomes a real correspondence when we select upon the straight line a definite origin or zero-point o and a definite unit of length for the measurement of segments.

Section III of his monograph then begins with a reminder of a familiar limitation of that correspondence [Dedekind, 1901, p. 9]:

> Of the greatest importance, however, is the fact that in the straight line L there are infinitely many points which correspond to no rational number.

In other words, the set of rational numbers is full of gaps, a property which Dedekind also called a state of "incompleteness" or "discontinuity." Noting that "we ascribe to the straight line completeness,

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[1]Dedekind was born on October 6, 1831, died February 12, 1958, Braunschweig, Germany. He earned his doctorate in mathematics at the University of Göttingen in 1852 under the supervision of Carl Friedrich Gauss (1777-1855).

absence of gaps, or continuity," Dedekind then asked the natural question: "In what then does this continuity [of the straight line] consist?" [Dedekind, 1901, p. 10]. His (perhaps obvious sounding) answer to this question provided the key idea behind his definition the irrational numbers. He found that the answer lies in the following principle [Dedekind, 1901, p. 11]:

> If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions.

In Sections IV through VI of his paper, Dedekind used this "severing" principle to create the irrational numbers and to outline the properties of the set of real numbers based on his new definition; the ideas in these section of his paper will be our primary focus in this project. In the final section of his paper, Section VII, was devoted to "infinitesimal analysis," where Dedekind discussed a couple of theorems on limits and proved them with the aid of cuts his newly-created purely-arithmetical definition of the real numbers.

Throughout this project you will read (translations of) portions of Dedekind's monograph. In the quoted text, you will see that his notation for the familiar sets of numbers are a bit different from what we use nowadays. For the set of rational numbers $\mathbb{Q}$, Dedekind employed R, whereas he designated the set of real numbers $\mathbb{R}$ by $\mathfrak{R}$. For our own work, we shall use modern set notation.

1 Dedekind Laments

Dedekind wanted to develop the idea of irrational numbers, purely from what is already know about the set of rational numbers and its properties.

> [The] way in which the irrational numbers are usually introduced is based directly upon the conception of extensive magnitudes—which itself is nowhere carefully defined—and explains number as the result of measuring such a magnitude by another of the same kind.[†] Instead of this I demand that arithmetic shall be developed out of itself.
>
> That such comparisons with non-arithmetic notions have furnished the immediate occasion for the extension of the number-concept may, in a general way, be granted (though this was certainly not the case in the introduction of complex numbers); but this surely is no sufficient ground for introducing these foreign notions into arithmetic, the science of numbers. Just as negative and fractional rational numbers are formed by a new creation, and as the laws of operating with these numbers must and can be reduced to the laws of operating with positive integers, so we must endeavor completely to define irrational numbers by means of the rational numbers alone. The question only remains how to do this.

[†]Dedekind's Footnote: The apparent advantage of the generality of this definition of number disappears as soon as we consider complex numbers. According to my view, on the other hand, the notion of the ratio between two numbers of the same kind can be clearly developed only after the introduction of irrational numbers.

$\boxed{\textbf{Task 1}}$ Answer the following in your own words: According to passage, how did Dedekind want to approach the development (or definition) of irrational numbers, and what was his frustration about the "usual" way in which irrational numbers are introduced? Do you agree or disagree with his view? Explain why or why not?

2 Dedekind Acts

As mentioned earlier, Dedekind recalled, in Section II of his text, a correspondence between the set of rational numbers to points of a straight line by fixing a particular point as the origin or *zero point* and a unit of length to measure the length of segments, that is to say, a particular length to measure length 1. Recognizing that the set or system of rational numbers has certain gaps that make it not completely comparable to a straight line, which he deemed to be a continuous system, Dedekind set out, beginning in his Section III, to establish or create new numbers to fill in these gaps so that, as he wrote, "the domain of numbers shall gain the same completeness, or as we may say at once, the same *continuity*, as the straight line" [Dedekind, 1901, p. 9].

2.1 Dedekind Cuts

Ironically, Dedekind began to fill the gaps by first making *cuts*.

> [It] is sufficiently obvious how the discontinuous domain R of rational numbers may be rendered complete so as to form a continuous domain. In Section I it was pointed out that every rational number a effects a separation of the system R into two classes such that every number a_1 of the first class A_1 is less than every number a_2 of the second class A_2; the number a is either the greatest number of the class A_1 or the least number of the class A_2. If now any separation of the system R into two classes A_1, A_2 is given which possesses only this characteristic property that every number a_1 in A_1 is less than every number a_2 in A_2, then for brevity we shall call such a separation a cut [ein Schnitt] and designate it by (A_1, A_2). We can then say that every rational number a produces one cut or, strictly speaking, two cuts, which, however, we shall not look upon as essentially different; this cut possesses, besides, the property that either among the numbers of the first class there exists a greatest or among the numbers of the second class a least number. And conversely, if a cut possesses this property, then it is produced by this greatest or least rational number.

As stated at the beginning of this passage, it is in Section I where Dedekind first introduced the idea of a separation of $\mathbb{Q}$ (R in Dedekind's text) into two classes A_1 and A_2 effected by a rational number a, such that every number in A_1 is less than every number in A_2. Moreover, in that earlier section, he said that the rational number a "may be assigned at pleasure to the first or second class, being respectively the greatest number of the first class or the least of the second" [Dedekind, 1901, p. 6].

Task 2 Reread the second sentence of the passage above. Now let $a = \frac{1}{2}$, $A_1 = \left\{x \in \mathbb{Q} : x \leq \frac{1}{2}\right\}$, $A_2 = \left\{x \in \mathbb{Q} : x > \frac{1}{2}\right\}$. Do A_1 and A_2 fit the description provided in that sentence, and, in particular, is every number a_1 in A_1 less than every number a_2 in A_2? Explain your reasoning.[2]

Task 3 Reflect on the following question: Do you like the word *cut* (in German, *Schnitt*) as a name for the separation of R into the classes A_1 and A_2 as described by Dedekind above? If so, why? If not, what other name might you suggest, and explain why you think it is a better fit?

Task 4 Besides requiring that A_1 and A_2 be subsets of $\mathbb{Q}$, what (only) other condition did Dedekind impose on the classes A_1 and A_2, so that (A_1, A_2) constitutes a cut? In particular, does a cut (A_1, A_2) require that either A_1 contain a largest rational number or that A_2 contains a smallest rational number?

Task 5 In the previous passage, Dedekind said that any rational number produces "two cuts, which, however, we shall not look upon as essentially different." Let $a \in \mathbb{Q}$. Use modern set notation to express sets A_1, A_2, B_1, and B_2 so that (A_1, A_2) and (B_1, B_2) are the two cuts that both correspond to a and therefore not essentially different. (Hint: In the case of $a = \frac{1}{2}$, the a possible option for sets A_1 and A_2 are provided in Task 2.)

Task 6 Decide which of the following pairs of A_1 and A_2 constitute a cut. If your answer is NO, explain why.

(a) $A_1 = \{x \in \mathbb{Q} : x < 0\}$, $A_2 = \{x \in \mathbb{Q} : x > 0\}$

(b) $A_1 = \mathbb{Z}$, $A_2 = \mathbb{Q} \setminus \mathbb{Z}$

(c) $A_1 = \{x \in \mathbb{Q} : x^3 < 5\}$, $A_2 = \{x \in \mathbb{Q} : x^3 \geq 5\}$

(d) $A_1 = \{x \in \mathbb{Q} : x^3 < 5\}$, $A_2 = \{x \in \mathbb{Q} : x^3 > 5\}$

2.2 Dedekind Darns

Immediately after the excerpt that we read in the previous section of this project, Dedekind made the point that a cut, as he defined it, does not have to be produced by a rational number in the way first introduced in Section I of this text. In the following passage, Dedekind described an example of such a cut.

[2]The modern set notation (using braces) that we use in this task had not yet been developed when Dedekind produced this work.

But it is easy to show that there exist infinitely many cuts not produced by rational numbers. The following example suggests itself most readily.

Let D be a positive integer but not the square of an integer, then there exists a positive integer λ such that

$$\lambda^2 < D < (\lambda+1)^2.$$

If we assign to the second class A_2, every positive rational number a_2 whose square is $> D$, to the first class A_1 all other rational numbers a_1, this separation forms a cut (A_1, A_2), i.e., every number a_1 is less than every number a_2. For if $a_1 = 0$, or is negative, then on that ground a_1 is less than any number a_2, because, by definition, this last is positive; if a_1 is positive, then is its square $\leqq D$, and hence a_1 is less than any positive number a_2 whose square is $> D$.

But this cut is produced by no rational number. To demonstrate this it must be shown first of all that there exists no rational number whose square $= D$. Although this is known from the first elements of the theory of numbers, still the following indirect proof may find place here. If there exist a rational number whose square $= D$, then there exist two positive integers t, u, that satisfy the equation

$$t^2 - Du^2 = 0$$

and we may assume that u is the *least* positive integer possessing the property that its square, by multiplication by D, may be converted into the square of an integer t. Since evidently

$$\lambda u < t < (\lambda+1)u,$$

the number $u' = t - \lambda u$ is a positive integer certainly less than u. If further we put $t' = Du - \lambda t$, t' is likewise a positive integer, and we have

$$t'^2 - Du'^2 = (\lambda^2 - D)(t^2 - Du^2) = 0,$$

which is contrary to the assumption respecting u. Hence the square of every rational number x is either $< D$ or $> D$.

Having concluded that the square of every rational number is either less than or greater than D. Dedekind went on to show that, in the cut (A_1, A_2) described in the previous passage, A_1 contains no greatest (rational) number nor does A_2 contain a least. Thus, no rational number produces this cut.

Task 7 It is often helpful to work through a proof by looking at a special case or specific example. We shall do this for Dedekind's argument in the previous passage.

(a) In Dedekind's argument above, let $D = 2$. Rewrite all the steps of the indirect proof until the end of the given passage. Does this argument show that $\sqrt{2}$ is irrational? Where does the contradiction come from? What steps in the proof are unclear or need further explanation?

(b) At some point in his argument above, Dedekind said, "...evidently, $\lambda u < t < (\lambda + 1)u$...". Is this line "evident" to you? What initial assumption does this come from? Hint: Work backwards, starting with dividing all parts of the inequality by u.

(c) A classic indirect argument for the irrationality of $\sqrt{2}$ involves starting with the assumption of the contrary, that is there exist positive integers p and q with no common factors, such that $\sqrt{2} = \frac{p}{q}$. Complete the proof and compare and contrast with the one you wrote for (a). In particular, where does the contradiction come from? Is it possible to apply the assumptions given in Dedekind's proof to this one?

At this point, Dedekind was ready to define an irrational number.

In this property that not all cuts are produced by rational numbers consists the incompleteness or discontinuity of the domain R of all rational numbers. Whenever, then, we have to do with a cut (A_1, A_2) produced by no rational number, we create a new, an *irrational* number, which we regard as completely defined by this cut (A_1, A_2); we shall say that the number corresponds to this cut, or that it produces this cut. From now on, therefore, to every definite cut there corresponds a definite rational or irrational number, and we regard two numbers as different or unequal always and only when they correspond to essentially different cuts.

Task 8 In your own words, how did Dedekind define (or "create") an *irrational* number?

3 Dedekind Stitches

Having described irrational numbers as those numbers whose cuts are not produced by any rational number, Dedekind's next task was to show how the combined system of rational and irrational numbers can be naturally imbued by properties analogous to those already established or obeyed by the rational numbers themselves. In doing so, Dedekind figuratively stitched together, all cuts, whether produced by a rational or an irrational number, to form the system of real numbers.

3.1 Dedekind Orders

In the second half of his Section IV, Dedekind examined the relation between any two cuts whether corresponding to a rational or an irrational number. He sought to establish an ordering of the numbers defined by such cuts based on the well-established ordering of rational numbers, the elements of the pairs of classes that constitute a cut.

In order to obtain a basis for the orderly arrangement of all *real*, i.e., of all rational and irrational numbers, we must investigate the relation between any two cuts (A_1, A_2) and (B_1, B_2) produced by any two numbers α and β. Obviously a cut (A_1, A_2) is given completely when one of the two classes, e.g., the first A_1 is known, because the second A_2 consists of all rational numbers not contained in A_1, and the characteristic property of such a first class lies in this that if the number a_1 is contained in it, it also contains all numbers less than a_1. If now we compare two such first classes A_1, B_1 with each other, it may happen

1. That they are perfectly identical, i.e., that every number contained in A_1 is also contained in B_1, and that every number contained in B_1 is also contained in A_1. In this case A_2 is necessarily identical with B_2, and the two cuts are perfectly identical, which we denote in symbols by $\alpha = \beta$ or $\beta = \alpha$.

But if the two classes A_1, B_1 are not identical, then there exists in the one, e.g., in A_1, a number $a'_1 = b'_2$ not contained in the other B_1 and consequently found in B_2; hence all numbers b_1 contained in B_1 are certainly less than this number $a'_1 = b'_2$ and therefore all numbers b_1 are contained in A_1.

2. If now this number a'_1 is the only one in A_1 that is not contained in B_1, then is every other number a_1 contained in A_1 also contained in B_1 and is consequently $< a'_1$, i. e., a'_1 is the greatest among all the numbers a_1, hence the cut (A_1, A_2) is produced by the rational number $\alpha = a'_1 = b'_2$. Concerning the other cut (B_1, B_2) we know already that all numbers b_1 in B_1 are also contained in A_1 and are less than the number $a'_1 = b'_2$ which is contained in B_2; every other number b_2 contained in B_2 must, however, be greater than b'_2, for otherwise it would be less than a_1, therefore contained in A_1 and hence in B_1; hence b'_2 is the least among all numbers contained in B_2, and consequently the cut (B_1, B_2) is produced by the same rational number $\beta = b'_2 = a'_1 = \alpha$. The two cuts are then only unessentially different.
3. If, however, there exist in A_1 at least two different numbers $a'_1 = b'_2$ and $a''_1 = b''_2$, which are not contained in B_1, then there exist infinitely many of them, because all the infinitely many numbers lying between a'_1 and a''_1 are obviously contained in A_1 (Section I, ii)[3] but not in B_1. In this case we say that the numbers α and β corresponding to these two essentially different cuts (A_1, A_2) and (B_1, B_2) are different, and further that α is *greater* than β, that β is *less* than α, which we express in symbols by $\alpha > \beta$ as well as $\beta < \alpha$. It is to be noticed that this definition coincides completely with the one given earlier, when α, β are rational.

[3] Section I, ii of Dedekind's text, not included here.

The remaining cases are these:

4. If there exists in B_1 one and only one number $b'_1 = a'_2$, that is not contained in A_1 then the two cuts (A_1, A_2) and (B_1, B_2) are only unessentially different and they are produced by one and the same rational number $\alpha = a'_2 = b'_1 = \beta$.
5. But if there are in B_1 at least two numbers which are not contained in A_1, then $\beta > \alpha$, $\alpha < \beta$.

Task 9 What did Dedekind mean by two cuts being "unessentially different" at the end of Case 2 (and again in Case 4)?

Task 10 Given cuts (A_1, A_2) and (B_1, B_2), where $A_1 = \{x \in \mathbb{Q} : x < -1\}$ and $B_2 = \{x \in \mathbb{Q} : x > -1\}$, determine A_2 and B_1 and write them out in set notation. What particular case do these cuts illustrate? What are α and β, which are the real numbers corresponding to the cuts (A_1, A_2) and (B_1, B_2), respectively?

Task 11 Given cuts (A_1, A_2) and (B_1, B_2), where $A_1 = \{x \in \mathbb{Q} : x \leq 10\}$ and $B_2 = \{x \in \mathbb{Q} : x > \pi\}$, determine A_2 and B_1 and write them out in set notation. What particular case do these cuts illustrate? What are α and β in this case?

3.2 Dedekind Continues

As we have seen in his Section IV, Dedekind established the system of real numbers as the union of rational and irrational numbers. This system is imbued with an order relation inherited from that of the rational numbers. Next, in Section V, Dedekind introduced the symbol $\mathfrak{R}$ (modern notation $\mathbb{R}$) to represent this system and presents four fundamental laws that make it "a well-arranged domain of one dimension"[Dedekind, 1901, p. 19]. It is worth noting Dedekind discussed these same four properties for points on a straight line back in his Section III. The first three can be summarized as follows:

I For any α, β, and γ in $\mathfrak{R}$, such that $\alpha > \beta$ and $\beta > \gamma$, then $\alpha > \gamma$. Here, β is said to *lie between* α and γ.

II There exist infinitely many real numbers lying between any two distinct ones.

III Any real number α induces a separation of the system $\mathfrak{R}$ into two classes $\mathfrak{A}_1$, $\mathfrak{A}_2$, both of which contain infinitely many numbers. The number α may be assigned to either class, so that if $\alpha_1 \in \mathfrak{A}_1$ and $\alpha_2 \in \mathfrak{A}_2$, then either $\alpha_1 \leq \alpha < \alpha_2$ or $\alpha_1 < \alpha \leq \alpha_2$.

[*Alternative statement for III*: Any real number α induces a separation of the system $\mathfrak{R}$ into two classes $\mathfrak{A}_1$, $\mathfrak{A}_2$, both of which contain infinitely many numbers. The number α may be assigned to either class, so that either $\mathfrak{A}_1 = \{x \in \mathfrak{R} : x \leq \alpha\}$ and $\mathfrak{A}_2 = \{x \in \mathfrak{R} : x > \alpha\}$, or $\mathfrak{A}_1 = \{x \in \mathfrak{R} : x < \alpha\}$ and $\mathfrak{A}_2 = \{x \in \mathfrak{R} : x \geq \alpha\}$.]

He omitted the proofs of the above for cuts "in order not to weary the reader" [Dedekind, 1901, p. 19]. Dedekind did prove the fourth and final property that he presented. Importantly, he referred to this property, which is the converse of III, as "continuity" (in German Stetigkeit). Here is his proof.

IV. If the system $\mathfrak{R}$ of all real numbers breaks up into two classes $\mathfrak{A}_1$, $\mathfrak{A}_2$ such that every number α_1 of the class $\mathfrak{A}_1$ is less than every number α_2 of the class $\mathfrak{A}_2$ then there exists one and only one number α by which this separation is produced.

Proof. By the separation or the cut of $\mathfrak{R}$ into $\mathfrak{A}_1$ and $\mathfrak{A}_2$ we obtain at the same time a cut (A_1, A_2) of the system R of all rational numbers which is defined by this that A_1 contains all rational numbers of the class $\mathfrak{A}_1$ and A_2 all other rational numbers, i.e., all rational numbers of the class $\mathfrak{A}_2$. Let α be the perfectly definite number which produces this cut (A_1, A_2). If β is number different from α, there are always infinitely many rational numbers c lying between α and β. If $\beta < \alpha$, then $c < \alpha$; hence c belongs to the class A_1 and consequently also to the class $\mathfrak{A}_1$, and since at the same time $\beta < c$ then β also belongs to the same class $\mathfrak{A}_1$, because every number in $\mathfrak{A}_2$ is greater than every number c in $\mathfrak{A}_1$. But if $\beta > \alpha$, then is $c > \alpha$; hence c belongs to the class A_2 and consequently also to the class $\mathfrak{A}_2$, and since at the same time $\beta > c$, then β also belongs to the same class $\mathfrak{A}_2$, because every number in $\mathfrak{A}_1$ is less than every number c in $\mathfrak{A}_2$. Hence every number β different from α belongs to the class $\mathfrak{A}_1$ or to the class $\mathfrak{A}_2$ according as $\beta < \alpha$ or $\beta > \alpha$; consequently α itself is either the greatest number in $\mathfrak{A}_1$ or the least number in $\mathfrak{A}_2$, i. e., α is one and obviously the only number by which the separation of $\mathfrak{R}$ into the classes $\mathfrak{A}_1$, $\mathfrak{A}_2$ is produced.[4] Which was to be proved.

Task 12 Go back through the proof and identify and mark the portions (either by underlining or highlighting) where Dedekind uses Properties I, II, III, respectively.

3.3 Dedekind Operates

Dedekind, in his Section VI, sought to address the question of operations among real numbers as well as other properties inherited from the system of rational numbers (such as its field properties and functions that can be applied to it). He stopped short of performing this monumental task and instead restricted himself to a thorough examination of operation of addition.

[4]Author's Footnote: A typographical error that appears in the translation has been corrected in this sentence of the excerpt. Instead of $\mathfrak{R}$ in the original German, the translation has R, but it should be the former

To reduce any operation with two real numbers α, β to operations with rational numbers, it is only necessary from the cuts (A_1, A_2), (B_1, B_2) produced by the numbers α and β in the system R to define the cut (C_1, C_2) which is to correspond to the result of the operation, γ. I confine myself here to the discussion of the simplest case, that of addition.

If c is any rational number, we put it into the class C_1, provided there are two numbers one a_1 in A_1 and one b_1 in B_1 such that their sum $a_1 + b_1 \geqq c$; all other rational numbers shall be put into the class C_2. This separation of all rational numbers into the two classes C_1, C_2 evidently forms a cut, since every number c_1 in C_1 is less than every number c_2 in C_2. If both α and β are rational, then every number c_1 contained in C_1 is $\leqq \alpha + \beta$, because $a_1 \leqq \alpha$, $b_1 \leqq \beta$, and therefore $a_1 + b_1 \leqq \alpha + \beta$; further, if there were contained in C_2 a number $c_2 < \alpha + \beta$, hence $\alpha + \beta = c_2 + p$, where p is a positive rational number, then we should have

$$c_2 = \left(\alpha - \frac{1}{2}p\right) + \left(\beta - \frac{1}{2}p\right),$$

which contradicts the definition of the number c_2, because $\alpha - \frac{1}{2}p$ is a number in A_1, and $\beta - \frac{1}{2}p$ is a number in B_1; consequently every number c_2 contained in C_2 is $\geqq \alpha + \beta$. Therefore in this case the cut (C_1, C_2) is produced by the sum $\alpha + \beta$. Thus we shall not violate the definition which holds in the arithmetic of rational numbers if in all cases we understand by the sum $\alpha + \beta$ of any two real numbers α, β that number γ by which the cut (C_1, C_2) is produced. Further, if only one of the two numbers α, β is rational, e.g., α, it is easy to see that it makes no difference with the sum $\gamma = \alpha + \beta$ whether the number α is put into the class A_1 or into the class A_2.

To reinforce your understanding of how Dedekind defined addition of two real numbers, complete the next two tasks.

Task 13 Suppose we have the following information about α and β with corresponding cuts (A_1, A_2) and (B_1, B_2) respectively:

- $\alpha = 0$, $A_1 = \{x \in \mathbb{Q} : x \leq 0\}$ and $A_2 = \{x \in \mathbb{Q} : x > 0\}$.
- $\beta = \pi$, $B_1 = \{x \in \mathbb{Q} : x < \pi\}$ and $B_2 = \{x \in \mathbb{Q} : x \geq \pi\}$.

Now let (C_1, C_2) be a candidate for a cut corresponding to the real number $\gamma = \alpha + \beta$.

(a) Should $c = 3.14$ be placed in C_1 or C_2? If C_1, name specific values for $a_1 \in A_1$ and $b_1 \in B_2$ such that $a_1 + b_1 \geqq c$. Why can't $c = 3.14$ be placed in the other?

(b) Should $c = 3.15$ be placed in C_1 or C_2?

Task 14 Suppose we have the following information about α and β with corresponding cuts (A_1, A_2) and (B_1, B_2) respectively:

- $\alpha = 0$, $A_1 = \{x \in \mathbb{Q} : x < 0\}$ and $A_2 = \{x \in \mathbb{Q} : x \geq 0\}$.
- $\beta = \pi$, $B_1 = \{x \in \mathbb{Q} : x < \pi\}$ and $B_2 = \{x \in \mathbb{Q} : x \geq \pi\}$.

Now let (C_1, C_2) be a candidate for a cut corresponding to the real number $\gamma = \alpha + \beta$.

(a) Should $c = 3.14$ be placed in C_1 or C_2? If C_1, name specific values for $a_1 \in A_1$ and $b_1 \in B_2$ such that $a_1 + b_1 \geqq c$. Why can't $c = 3.14$ be placed in the other?

(b) Should $c = 3.15$ be placed in C_1 or C_2?

Task 15 At this point take a moment to write a reflection on the following question: How successful do you think Dedekind has been in obtaining his goal of defining the irrationals solely in terms of the rationals in a way that "fills in the gaps."

In his text, after his thorough discussion of addition, Dedekind merely stated that other arithmetic operations can be defined, but did not provide a full exposition. The following task challenges you to define the other major operation performed on the real numbers: multiplication.

Task 16 How would you define *multiplication* of real numbers using Dedekind cuts? Consider the following steps:

1. Given two real numbers α and β, with corresponding respective cuts (A_1, A_2) and (B_1, B_2), how would you define the cut (C_1, C_2) corresponding to γ, the number resulting from this operation of multiplication?
2. If c is a rational number, how would you decide whether c belongs to C_1 or C_2?
3. How would particularities such as "a negative number times another negative number produces a positive number" or "a negative number times a positive number results in a negative number" be resolved in your definition of multiplication when performed using Dedekind cuts?

3.4 Dedekind Challenges

As in the case of multiplication, Dedekind did not provide much detail in establishing the panoply of properties that operations among real numbers possess. These parting tasks challenge you to attempt a few others.

Task 17 After completing the previous three tasks, involving addition and multiplication of Dedekind cuts, a natural way to proceed is to establish the distributive law for real numbers, $a \cdot (b + c) = a \cdot b + a \cdot c$. This entails showing that any cut determined by the left side of the equation is identical, or as Dedekind put it *unessentially different* from the one determined by the right side.[5]

[5] Refer to [Dedekind, 1901, pp. 13, 15, and 17] or the quoted texts in 2.1, 2.2, and 3.1 for Dedekind's explanation of essentially different and unessentially different cuts.

Task 18 For a further challenge, you might choose to prove some of your other favorite properties of the real numbers. For example, how would the square root of a number be defined in terms of cuts? The first thing to consider is the fact that the square root of a (nonnegative) *rational number* is not necessarily rational (e.g. $\sqrt{2}$; see Task 7, part (a)). Could you establish, for instance, the property $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$?

A word of advice: For this task, and even for the earlier ones, there is no shame in first considering particular cases or examples, before attempting to prove something in its full generality; it is, in fact, encouraged. Dedekind, himself, chose the example $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ as a representative of this property in his text. Feel free to get started with this example.

Dedekind was certainly not the first mathematician to demand rigor, clarity, and generality in mathematical thought and practice, nor will he be the last. Among his contemporaries, Georg Cantor (1845-1918), Eduard Heine (1821-1881), and Karl Weierstrass (1815-1897) all pursued the rigorous development of the theories of the real numbers, using the set of rational numbers as a starting point.[6] This pursuit of the *arithmetization of real analysis* that occurred in the latter half of the nineteenth century, with its emphasis in clear definitions and foundations, has certainly had the lasting effect in the way we currently teach, study, and pursue research in modern mathematics and train future mathematicians.

References

R. Dedekind. *Stetigkeit und Irrationale Zahlen.* Friedrich Vieweg und Sohn, Braunschweig, 1872.

R. Dedekind. *Essays on the Theory of Numbers.* The Open Court Pub. Co., Chicago, 1901. Translated from the German by W. W. Beman.

[6]For a concise exposition of these approaches, see the article by Joanne Snow, "Views on the Real Numbers and the Continuum," *The Review of Modern Logic*, Volume 9, Numbers 1 & 2 (November 2001–November 2003) [Issue 29], pp. 95–113.

Notes to Instructors

PSP Content: Topics and Goals

The primary content learning outcome of this PSP is an understanding of the rigorous development of the set of real numbers from the properties of the rationals as conceived by Richard Dedekind. Therefore, this project fits naturally in an introduction to real analysis course for instructors who wish not to skimp on this topic, which is often considered optional in such courses. The project can also be implemented in an introduction to proofs course, in particular, for such courses with a focus on the development of number systems. Therefore, a complementary goal of this project is to hone the logical reasoning skills necessary to formulate and write proofs. In particular, students practice and refine their ability to interpret and apply abstract definitions within a proof and gain an appreciation for making use of a careful approach in mathematical methodology.

It is not unusual for mathematics students at the college level to gain familiarity with the development of the number systems from the natural numbers, to the integers, and then to the rationals. But the development or construction of the real numbers from the rationals is often overlooked. However, it could be challenging to approach or tackle such a topic that seems to be already *familiar* to students. After all, they will be asked to develop operations and properties of real numbers (e.g. addition, multiplication, the distributive property) that they have been using for much of their mathematical life. Nevertheless, this project aims to reveal to the students an often overlooked topic and fill in a gap in the development of the major number systems.

Student Prerequisites

Regardless of the course in which this PSP is used, students' prior understanding of operations among real/rational numbers and their order properties is a sufficient prerequisite. In courses that formally study field and order properties, completing this PSP would fit naturally afterwards. In addition, students should have some working knowledge of modern set notation and symbols and some initiation with reading and writing proofs.

PSP Design, and Task Commentary

After a brief introduction focused on the genesis of the primary source text by Dedekind, the PSP is divided into three major sections. The first section consists of a brief excerpt expressing Dedekind's dissatisfaction with the way irrational numbers lack the rigorous development necessary to fully integrate in the analysis of numbers. The only task in this section asks the students to give a brief interpretation of the excerpt. Section 2 is devoted to the definition of cuts with the aim to define irrational numbers (and hence, all real numbers) using them. The tasks in this subsection should help students understand the structure of cuts through examples. While Dedekind initially used rational numbers to motivate the definition of cuts, he also showed that there exist cuts that are not *produced* by rationals. Task 6 is also designed to understand Dedekind's claim through an example (that of $\sqrt{2}$).

Section 3 is by far the longest section. This is where the students learn how Dedekind developed the order properties and addition operation of real numbers based on his formal definition of cuts. Once more, the student tasks serve to interpret Dedekind's exposition through examples. The final tasks in the latter part of this section challenge the students to go beyond what Dedekind chose explicitly to develop in his text. In Task 16, the students are asked to define and thoroughly explain *multiplication* of real numbers as performed on cuts, something that Dedekind omitted from his monograph. Tasks 17 and 18 challenge the students even more.

Many tasks in the PSP require students to do a variety of mathematical exercises. Others ask the students to express some of Dedekind's statements in their own words or solicit an opinion. For example, Task 3 asks the students whether they like the choice of the word *cut*, or if they can think of the another word that might be appropriate to use. This could potentially lead to a lively discussion on mathematical terminology. Task 15 demands a deeper reflection as it asks the students to assess Dedekind's success in rigorously defining irrational numbers. Task 12 is quite unique in that it requires the students to mark a portion of Dedekind's text (a proof). To assign this task, the instructor may opt to provide the students a supplementary copy of the text to mark, or students can mark their original copies and use a device such as a photo scanner to turn in their work.

Suggestions for Classroom Implementation

In implementing this project, the author believes that students benefit from a mix of individual work and preparation, working within small groups, and occasionally an entire class discussion. Individual instructors should naturally adjust according to their own strengths and preferences. The first five tasks and the reading that goes with them can be assigned as pre-preparation, so that during the the first class session the students (as an entire class or in small groups) can compare their answers with each other. The last two tasks are intended to be a challenge, and as such, they can be assigned as individual homework, or something that small groups of students work on during class, for instance, while sitting at a table while working on the same document or working by a chalk or whiteboard.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50-minute class period)

To implement the project using three 50-minute class periods, we suggest the following schedule:

- **Preparation for Day 1**: Assign the introduction, all of Section 1, up to Section 2.1, and Tasks 1–5 as homework.

- **Day 1**: The first 5–10 minutes can be used to discuss and compare answers from the preparation homework for Day 1. Then, students read Section 2.2 and work on Task 6 in class in small groups. Reading the source text at the end of Section 2.2 and Task 7 can be assigned as a parting exercise (last 5 minutes of class), or this can be part of homework.

- **Preparation for Day 2**: Assign the source text reading in Section 3.2 and Tasks 7-10 as homework.

- **Day 2**: The first 10-15 minutes can be used to discuss and compare answers from the preparation homework for Day 2. For the rest of the period the students work on Sections 3.3 and 3.4 with the goal of at least starting Task 16.

- **Preparation for Day 3**: Ask the students to complete the unfinished tasks from Day 2 as homework, which is most likely Task 16.

- **Day 3**: For the third day, students can work on either Task 17 or Task 18 or both. (Or have half of the class work on Task 17 and the other half work on Task 18.)

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students. Estimates on the high end of the range assume most PSP work is completed by students working in small groups during class time. One of many variations possible would be to have students work on Tasks 6 to 16 in class spread over two or three days and to assign the challenge tasks, Task 17 and 18 (either one or both), as individual homework.

Connections to other Primary Source Projects

Additional PSPs that are suitable for use in introductory real analysis courses include the following; the PSP author name for each is listed parenthetically, along with the project topic if this is not evident from the PSP title. "Mini-PSPs," designed to be completed in 1–2 class periods, are designated with an asterisk (*).

- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis** (Janet Barnett)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/1/`
- *Topology from Analysis** (Nick Scoville)
  Also suitable for use in a course on topology.
  `https://digitalcommons.ursinus.edu/triumphs_topology/1/`
- *Investigations into Bolzano's Bounded Set Theorem* (Dave Ruch)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/14/`
- *Bolzano's Definition of Continuity, his Bounded Set Theorem, and an Application to Continuous Functions* (Dave Ruch)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/13`
- *An Introduction to a Rigorous Definition of Derivative* (Dave Ruch)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/7`
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real Analysis* (Derivatives and Intermediate Value Property; Janet Barnett)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/10/`
- *The Mean Value Theorem*(Dave Ruch)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/5/`
- *The Definite Integrals of Cauchy and Riemann* (Dave Ruch)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/11/`
- *Henri Lebesgue and the Development of the Integral Concept** (Janet Barnett)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/2/`
- *Investigations Into d'Alembert's Definition of Limit** (sequences; Dave Ruch)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/13/`
- *Euler's Rediscovery of e** (Dave Ruch)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/3/`

- *Abel and Cauchy on a Rigorous Approach to Infinite Series* (Dave Ruch)
  https://digitalcommons.ursinus.edu/triumphs_analysis/4/
- *The Cantor set before Cantor** (Nick Scoville)
  Also suitable for use in a course on topology.
  https://digitalcommons.ursinus.edu/triumphs_topology/2/

Dedekind's emphasis on abstraction, his continual quest for generality and his careful methodology also feature in the following two PSPs, both based on his ground-breaking work on ideals. These PSPs are designed for use in Number Theory and Abstract Algebra respectively; the second of these could also be used in an Introduction to Proof course.

- Dedekind and the Creation of Ideals (Janet Barnett)
  https://digitalcommons.ursinus.edu/triumphs_abstract/1/
- Gaussian Integers and Dedekind Ideals: A Number Theory Project (Janet Barnett)
  https://digitalcommons.ursinus.edu/triumphs_number3/

Additional PSPs that are appropriate for use in an Introduction to Proofs course include the following:

- Greatest Common Divisor: Algorithm and Proof (Mary Flagg)
  https://digitalcommons.ursinus.edu/triumphs_number/10/
- A Look at Desargues' Theorem from Dual Perspectives (Carl Lienert)
  https://digitalcommons.ursinus.edu/triumphs_geometry/3/

Acknowledgments

The development of this student project has been partially supported by the TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) Program with funding from the National Science Foundation's Improving Undergraduate STEM Education Program under Grant Nos. 1523494, 1523561, 1523747, 1523753, 1523898, 1524065, and 1524098. Any opinions, findings, and conclusions or recommendations expressed in this project are those of the author and do not necessarily represent the views of the National Science Foundation.

The author thanks the TRIUMPHS team for organizing the very beneficial PSP discussion series that took place in the summer of 2020. He also wishes to express particular gratitude to David Pengelley, for his insights and beneficial suggestions during the writing of the PSP, as well as to Janet Barnett and Dominic Klyve, for their thorough and careful editing during the review and revision process.

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